Title

> **mca —** Multiple and joint correspondence analysis

Description

mca performs multiple correspondence analysis (MCA) or joint correspondence analysis (JCA) on a series of categorical variables. MCA and JCA are two generalizations of correspondence analysis (CA) of a cross-tabulation of two variables (see [MV] **ca**) to the cross-tabulation of multiple variables.

mca performs an analysis of two or more integer-valued variables. Crossing (also called stacking) of integer-valued variables is also allowed.

Quick start

Multiple and joint correspondence analysis

Correspondence analysis of the multiway cross-tabulation of catvar1, catvar2, and catvar3
    mca catvar1 catvar2 catvar3

Also produce the MCA coordinate plot
    mca catvar1 catvar2 catvar3, plot

Use 3 dimensions instead of 2, and display the tables compactly
    mca catvar1 catvar2 catvar3, dimensions(3) compact

Use principal normalization instead of the default standard normalization
    mca catvar1 catvar2 catvar3, normalize(principal)

Joint correspondence analysis of catvar1, catvar2, and catvar3
    mca catvar1 catvar2 catvar3, method(joint)

Multiple and joint correspondence analysis with crossed (stacked) variables

Multiple correspondence analysis of the multiway tabulation of catvar1, catvar2 and the crossed variables catvar3 and catvar4 with the crossed variables called z
    mca catvar1 catvar2 (z: catvar3 catvar4)

Same as above, but joint correspondence analysis
    mca catvar1 catvar2 (z: catvar3 catvar4), method(joint)

Menu

Statistics > Multivariate analysis > Correspondence analysis > Multiple correspondence analysis (MCA)

Syntax

Basic syntax for two or more categorical variables

> mca *varlist* [*if*] [*in*] [*weight*] [, *options*]

Full syntax for use with two or more categorical or crossed (stacked) categorical variables

> mca *speclist* [*if*] [*in*] [*weight*] [, *options*]

where

> *speclist* = *spec* [*spec* ...]
>
> *spec* = *varlist* | (*newvar* : *varlist*)

options	Description
Model	
supplementary(*speclist*)	supplementary (passive) variables
method(burt)	use the Burt matrix approach to MCA; the default
method(indicator)	use the indicator matrix approach to MCA
method(joint)	perform a joint correspondence analysis (JCA)
dimensions(*#*)	number of dimensions (factors, axes); default is dim(2)
normalize(standard)	display standard coordinates; the default
normalize(principal)	display principal coordinates
iterate(*#*)	maximum number of method(joint) iterations; default is iterate(250)
tolerance(*#*)	tolerance for method(joint) convergence criterion; default is tolerance(1e-5)
missing	treat missing values as ordinary values
noadjust	suppress the adjustment of eigenvalues (method(burt) only)
Codes	
report(variables)	report coding of crossing variables
report(crossed)	report coding of crossed variables
report(all)	report coding of crossing and crossed variables
length(min)	use minimal length unique codes of crossing variables
length(*#*)	use # as coding length of crossing variables
Reporting	
ddimensions(*#*)	display # singular values; default is ddim(.) (all)
points(*varlist*)	display tables for listed variables; default is all variables
compact	display statistics table in a compact format
log	display the iteration log (method(joint) only)
plot	plot the coordinates (that is, mcaplot)
maxlength(*#*)	maximum number of characters for labels in plot; default is maxlength(12)

`bootstrap`, `by`, `collect`, `jackknife`, `rolling`, and `statsby` are allowed with `mca`; see [U] **11.1.10 Prefix commands**. However, `bootstrap` and `jackknife` results should be interpreted with caution; identification of the `mca` parameters involves data-dependent restrictions, possibly leading to badly biased and overdispersed estimates (Milan and Whittaker 1995).

Weights are not allowed with the `bootstrap` prefix; see [R] **bootstrap**.

`fweight`s are allowed; see [U] **11.1.6 weight**.

See [U] **20 Estimation and postestimation commands** for more capabilities of estimation commands.

Options

> [Model]

`supplementary`(*speclist*) specifies that *speclist* are supplementary variables. Such variables do not affect the MCA solution, but their categories are mapped into the solution space. For `method(indicator)`, this mapping uses the first method described by Greenacre (2006). For `method(burt)` and `method(joint)`, the second and recommended method described by Greenacre (2006) is used, in which supplementary column principal coordinates are derived as a weighted average of the standard row coordinates, weighted by the supplementary profile. See the syntax diagram for the syntax of *speclist*.

`method`(*method*) specifies the method of MCA/JCA.

> `method(burt)`, the default, specifies MCA, a categorical variables analogue to principal component analysis (see [MV] **pca**). The Burt method performs a CA of the Burt matrix, a matrix of the two-way cross-tabulations of all pairs of variables.

> `method(indicator)` specifies MCA via a CA on the indicator matrix formed from the variables.

> `method(joint)` specifies JCA, a categorical variables analogue to factor analysis (see [MV] **factor**). This method analyzes a variant of the Burt matrix, in which the diagonal blocks are iteratively adjusted for the poor diagonal fit of MCA.

`dimensions`(*#*) specifies the number of dimensions (= factors = axes) to be extracted. The default is `dimensions(2)`. If you specify `dimensions(1)`, the categories are placed on one dimension. The number of dimensions is no larger than the number of categories in the active variables (regular and crossed) minus the number of active variables, and it can be less. This excludes supplementary variables. Specifying a larger number than the dimensions available results in extracting all dimensions.

> MCA is a hierarchical method so that extracting more dimensions does not affect the coordinates and decomposition of inertia of dimensions already included. The percentages of inertia accounting for the dimensions are in decreasing order as indicated by the singular values. The first dimension accounts for the most inertia, followed by the second dimension, and then the third dimension, etc.

`normalize`(*normalization*) specifies the normalization method, that is, how the row and column coordinates are obtained from the singular vectors and singular values of the matrix of standardized residuals.

> `normalize(standard)` specifies that coordinates are returned in standard normalization (singular values divided by the square root of the mass). This is the default.

> `normalize(principal)` specifies that coordinates are returned in principal normalization. Principal coordinates are standard coordinates multiplied by the square root of the corresponding principal inertia.

`iterate`(*#*) is a technical and rarely used option specifying the maximum number of iterations. `iterate()` is permitted only with `method(joint)`. The default is `iterate(250)`.

`tolerance(#)` is a technical and rarely used option specifying the tolerance for subsequent modification of the diagonal blocks of the Burt matrix. `tolerance()` is permitted only with `method(joint)`. The default is `tolerance(1e-5)`.

`missing` treats missing values as ordinary values to be included in the analysis. Observations with missing values are omitted from the analysis by default.

`noadjust` suppresses principal inertia adjustment and is allowed with `method(burt)` only. By default, the principal inertias (eigenvalues of the Burt matrix) are adjusted. The unmodified principal inertias present a pessimistic measure of fit because MCA fits the diagonal of $\mathbf{P}$ poorly (see Greenacre [1984]).

_____| Codes |_____

`report(opt)` displays coding information for the crossing variables, crossed variables, or both. `report()` is ignored if you do not specify at least one crossed variable.

   `report(variables)` displays the coding schemes of the crossing variables, that is, the variables used to define the crossed variables.

   `report(crossed)` displays a table explaining the value labels of the crossed variables.

   `report(all)` displays the codings of the crossing and crossed variables.

`length(opt)` specifies the coding length of crossing variables.

   `length(min)` specifies that the minimal-length unique codes of crossing variables be used.

   `length(#)` specifies that the coding length # of crossing variables be used, where # must be between 4 and 32.

_____| Reporting |_____

`ddimensions(#)` specifies the number of singular values to be displayed. If `ddimensions()` is greater than the number of singular values, all the singular values will be displayed. The default is `ddimensions(.)`, meaning all singular values.

`points(varlist)` indicates the variables to be included in the tables. By default, tables are displayed for all variables. Regular categorical variables, crossed variables, and supplementary variables may be specified in `points()`.

`compact` specifies that point statistics tables be displayed multiplied by 1,000, enabling the display of more columns without wrapping output. The compact tables can be displayed without wrapping for models with two dimensions at line size 79 and with three dimensions at line size 99.

`log` displays an iteration log. This option is permitted with `method(joint)` only.

`plot` displays a plot of the row and column coordinates in two dimensions. Use `mcaplot` directly to select different plotting points or for other graph refinements; see [MV] **mca postestimation plots**.

`maxlength(#)` specifies the maximum number of characters for labels in plots. The default is `maxlength(12)`. # must be less than 32.

   Note: The reporting options may be specified during estimation or replay.

Remarks and examples

Remarks are presented under the following headings:

Introduction

Multiple correspondence analysis (MCA) and joint correspondence analysis (JCA) are methods for analyzing observations on categorical variables. MCA is usually viewed as an extension of simple correspondence analysis (CA) to more than two variables. CA analyzes a two-way contingency table; MCA and JCA analyze a multiway table.

MCA can be viewed as a generalization of principal component analysis where the variables to be analyzed are categorical, not continuous. MCA performs a CA on a Burt or indicator matrix; it explores the relationships within a set of variables, whereas CA has more focus on exploring relationships between two sets of variables. JCA is an extension of MCA that attempts to remedy discrepancies between CA and MCA.

For an introduction to MCA via CA, see Rencher and Christensen (2012) or Everitt and Dunn (2001). For an advanced introduction to MCA without previous discussion of CA, see Gower and Hand (1996). Greenacre (2006) approaches MCA from CA, then JCA from MCA, and gives a more advanced treatment. [MV] **ca** also introduces MCA concepts. Gower (1990) explores MCA history.

Three methods are implemented here. We will start with the simplest and most intuitive representation of unordered categorical data: the indicator matrix, usually denoted $\mathbf{Z}$, a matrix of zeros and ones with columns for all categories of all variables and rows corresponding to observations. A value of one indicates that a category is observed; a zero indicates that it is not. MCA can be performed as a CA on the indicator matrix; an equivalent to this method is method(indicator) with mca.

Here is a manufactured indicator matrix $\mathbf{Z}$, with 10 observations on three categorical variables, $w, x,$ and y, each with two categories indicated by $w_1, w_2, x_1, x_2, y_1, y_2$:

$$
\mathbf{Z} = \begin{array}{c}
\text{obs} \\
1. \\
2. \\
3. \\
4. \\
5. \\
6. \\
7. \\
8. \\
9. \\
10.
\end{array}
\begin{array}{cccccc}
w_1 & w_2 & x_1 & x_2 & y_1 & y_2 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0
\end{array}
$$

For large datasets with many variables and observations, the indicator matrix is burdensome and can exceed memory limitations. The Burt matrix, usually denoted as $\mathbf{B}$, is a cross-tabulation of all categories of all variables. $\mathbf{B} = \mathbf{Z}'\mathbf{Z}$. The Burt matrix is smaller than the indicator matrix $\mathbf{Z}$. Performing CA on the Burt matrix and performing CA on the indicator matrix are related but not the same: they give equivalent standard coordinates, but the total principal inertias from the indicator

matrix approach and Burt matrix approach are different. Performing a CA of the Burt matrix without further calculation is `method(burt)` along with the `noadjust` option.

The Burt matrix, $\mathbf{B}$, corresponding to matrix $\mathbf{Z}$, above:

$$
\mathbf{B} = \begin{array}{c} \\ w_1 \\ w_2 \\ x_1 \\ x_2 \\ y_1 \\ y_2 \end{array} \begin{array}{cccccc} w_1 & w_2 & x_1 & x_2 & y_1 & y_2 \\ \left(\begin{array}{cccccc} 7 & 0 & 4 & 3 & 3 & 4 \\ 0 & 3 & 1 & 2 & 3 & 0 \\ 4 & 1 & 5 & 0 & 3 & 2 \\ 3 & 2 & 0 & 5 & 3 & 2 \\ 3 & 3 & 3 & 3 & 6 & 0 \\ 4 & 0 & 2 & 2 & 0 & 4 \end{array} \right) \end{array}
$$

Applying CA to the Burt matrix or indicator matrix artificially inflates the χ^2 distances between profiles and the total inertia; see Gower (2006). This can be partially remedied after CA of the Burt matrix by scale readjustments of the MCA solution. Performing adjustments after a CA of the Burt matrix is the default, that is, if no method is specified `method(burt)` is assumed; unadjusted estimates may be obtained by using the option `noadjust`.

The third method, JCA, generalizes CA more naturally to more than two categorical variables. JCA attempts to account for the variation in the off-diagonal submatrices of the Burt matrix. It corrects the inflation of the total inertia. The solution can no longer be obtained by one application of the singular value decomposition (as in CA). JCA is obtained by specifying the option `method(joint)` to `mca`.

Compare MCA on two variables and CA

We illustrate MCA with a dataset from the International Social Survey Program on environment (ISSP 1993). This example is used in the MCA literature; see Greenacre (2006). We will look at the questions on attitudes toward science. We use data from the West German sample only and remove all observations containing missing data; 871 observations remain. The survey questions are

How much do you agree or disagree with each of these statements?

    A. We believe too often in science, and not enough in feelings and faith.

    B. Overall, modern science does more harm than good.

    C. Any change humans cause in nature—no matter how scientific—is likely to make things worse.

    D. Modern science will solve our environmental problems with little change to our way of life.

Each question has five possible response categories:

    1. Agree strongly

    2. Agree

    3. Neither agree nor disagree

    4. Disagree

    5. Disagree strongly

Supplementary demographic information is also available as coded categorical variables:

    Sex: male, female

    Age: (six groups) 16–24, 25–34, 35–44, 45–54, 55–64, 65 and older

    Education: (six groups) primary incomplete, primary completed, secondary incomplete, secondary completed, tertiary incomplete, tertiary completed.

▷ Example 1

Here is a summary of these data.

```
. use https://www.stata-press.com/data/r18/issp93
(Selection from ISSP (1993))

. describe

Contains data from https://www.stata-press.com/data/r18/issp93.dta
 Observations:            871                  Selection from ISSP (1993)
    Variables:              8                  17 May 2022 09:36
                                               (_dta has notes)

Variable        Storage   Display    Value
    name           type    format    label      Variable label

id               int      %9.0g                 Respondent identifier
A                byte     %26.0g     agree5      Too much science, not enough
                                                   feelings & faith
B                byte     %26.0g     agree5      Science does more harm than good
C                byte     %26.0g     agree5      Any change makes nature worse
D                byte     %26.0g     agree5      Science will solve environmental
                                                   problems
sex              byte     %9.0g      sex         Sex
age              byte     %9.0g      age         Age (6 categories)
edu              byte     %20.0g     edu         Education (6 categories)

Sorted by:
```

We begin by comparing MCA with two variables to CA with two variables. The default MCA analysis is a CA of the Burt matrix, performed with adjustment to the principal inertias. The unadjusted results from MCA will give different principal inertias from the CA, although the standard coordinates would

be the same. With adjustment, results are nearly identical. For a detailed discussion of the output of ca and mca, see [MV] **ca**.

```
. mca A B
Multiple/joint correspondence analysis         Number of obs    =         871
                                                Total inertia    =   .23775345
       Method: Burt/adjusted inertias           Number of axes   =           2
```

Dimension	Principal inertia	Percent	Cumul. percent
Dim 1	.1686131	70.92	70.92
Dim 2	.0586785	24.68	95.60
Dim 3	.010444	4.39	99.99
Dim 4	.0000178	0.01	100.00
Total	.2377535	100.00	

Statistics for column categories in standard normalization

Categories	Mass	Overall Quality	%inert	Dimension_1 Coord	Sqcorr	Contrib
A						
Agree stro~y	0.068	1.000	0.100	1.017	0.502	0.071
Agree	0.185	0.997	0.042	0.560	0.982	0.058
Neither ag~e	0.117	0.930	0.062	0.248	0.083	0.007
Disagree	0.102	0.922	0.123	-1.239	0.907	0.157
Disagree s~y	0.028	0.954	0.174	-2.741	0.845	0.207
B						
Agree stro~y	0.041	0.982	0.146	1.571	0.490	0.101
Agree	0.100	0.962	0.034	0.667	0.932	0.044
Neither ag~e	0.118	0.840	0.043	0.606	0.716	0.043
Disagree	0.161	0.769	0.043	-0.293	0.228	0.014
Disagree s~y	0.080	0.994	0.235	-1.926	0.900	0.298

Categories	Dimension_2 Coord	Sqcorr	Contrib
A			
Agree stro~y	1.718	0.498	0.202
Agree	0.116	0.015	0.002
Neither ag~e	-1.344	0.847	0.212
Disagree	-0.268	0.015	0.007
Disagree s~y	1.672	0.109	0.077
B			
Agree stro~y	2.671	0.493	0.291
Agree	-0.201	0.029	0.004
Neither ag~e	-0.427	0.124	0.022
Disagree	-0.764	0.541	0.094
Disagree s~y	1.055	0.094	0.089

With ca we use the norm(standard) option to obtain results in the same normalization as the preceding mca. [MV] **ca** discusses the normalize() option; mca has some of the normalize() options of ca.

The top table in the output for both ca and mca reports the principal inertias. The principal inertias reported by ca are simply the squares of the singular values. Two pieces of information reported by ca that are not reported by mca are the χ^2 value in the table and the explained inertia in the header. The χ^2 value is taken as a measure of dissimilarity between the row and column profiles

of the two-way table. χ^2 distance makes sense for a two-way table but has less justification when applied to the rows and columns of either the indicator matrix or the Burt matrix (Greenacre 2006). The explained inertia is the value from the cumulative percent column in the top table corresponding to the number of dimensions selected; it is simply not reported in the mca header.

The coordinates reported in the second table are the same. The mass, inertia, and contributions are half as large in mca as they are in ca; in ca the row and column each sum to the principal inertia, but in mca there are only columns, which sum to the principal inertia.

```
. ca A B, norm(standard)
```

Correspondence analysis

Number of obs	=	871
Pearson chi2(16)	=	207.08
Prob > chi2	=	0.0000
Total inertia	=	0.2378

5 active rows   Number of dim. = 2
5 active columns   Expl. inertia (%) = 95.60

Dimension	Singular value	Principal inertia	chi2	Percent	Cumul. percent
Dim 1	.4106252	.1686131	146.86	70.92	70.92
Dim 2	.2422364	.0586785	51.11	24.68	95.60
Dim 3	.1021961	.010444	9.10	4.39	99.99
Dim 4	.0042238	.0000178	0.02	0.01	100.00
Total		.2377535	207.08	100	

Statistics for row and column categories in standard normalization

		Overall		Dimension_1		
Categories	Mass	Quality	%inert	Coord	Sqcorr	Contrib
A						
Agree stro~y	0.137	1.000	0.200	1.017	0.502	0.141
Agree	0.370	0.997	0.084	0.560	0.982	0.116
Neither ag~e	0.234	0.930	0.123	0.248	0.083	0.014
Disagree	0.204	0.922	0.245	-1.239	0.907	0.314
Disagree s~y	0.055	0.954	0.348	-2.741	0.845	0.414
B						
Agree stro~y	0.082	0.982	0.291	1.571	0.490	0.201
Agree	0.200	0.962	0.068	0.667	0.932	0.089
Neither ag~e	0.235	0.840	0.086	0.606	0.716	0.087
Disagree	0.323	0.769	0.086	-0.293	0.228	0.028
Disagree s~y	0.161	0.994	0.470	-1.926	0.900	0.596

	Dimension_2		
Categories	Coord	Sqcorr	Contrib
A			
Agree stro~y	1.718	0.498	0.403
Agree	0.116	0.015	0.005
Neither ag~e	-1.344	0.847	0.423
Disagree	-0.268	0.015	0.015
Disagree s~y	1.672	0.109	0.154
B			
Agree stro~y	2.671	0.493	0.582
Agree	-0.201	0.029	0.008
Neither ag~e	-0.427	0.124	0.043
Disagree	-0.764	0.541	0.188
Disagree s~y	1.055	0.094	0.179

◁

MCA on four variables

Now we will take a look at MCA with more than two variables and at the different methods of performing MCA or JCA.

▷ Example 2

We continue to use the ISSP (1993) dataset, looking at all four questions on attitudes toward science. We use the default method of MCA, which is a CA of the Burt matrix for the data, followed by simple scale adjustments. We choose the principal normalization normalize(principal), which scales the coordinates by the principal inertias. MCA with the Burt matrix and adjustments explains at least 79.1% of the total inertia in the first two dimensions.

```
. mca A-D, normalize(principal)
```

Multiple/joint correspondence analysis   Number of obs   =   871
                                               Total inertia   = .17024552
   Method: Burt/adjusted inertias   Number of axes   =   2

Dimension	Principal inertia	Percent	Cumul. percent
Dim 1	.0764553	44.91	44.91
Dim 2	.0582198	34.20	79.11
Dim 3	.009197	5.40	84.51
Dim 4	.0056697	3.33	87.84
Dim 5	.0011719	0.69	88.53
Dim 6	6.61e-06	0.00	88.53
Total	.1702455	100.00	

Statistics for column categories in principal normalization

Categories	Mass	Overall Quality	%inert	Dimension_1 Coord	Sqcorr	Contrib
A						
Agree stro~y	0.034	0.963	0.060	0.508	0.860	0.115
Agree	0.092	0.659	0.023	0.151	0.546	0.028
Neither ag~e	0.059	0.929	0.037	-0.124	0.143	0.012
Disagree	0.051	0.798	0.051	-0.322	0.612	0.069
Disagree s~y	0.014	0.799	0.067	-0.552	0.369	0.055
B						
Agree stro~y	0.020	0.911	0.100	0.809	0.781	0.174
Agree	0.050	0.631	0.027	0.177	0.346	0.021
Neither ag~e	0.059	0.806	0.027	0.096	0.117	0.007
Disagree	0.081	0.620	0.033	-0.197	0.555	0.041
Disagree s~y	0.040	0.810	0.116	-0.374	0.285	0.074
C						
Agree stro~y	0.044	0.847	0.122	0.597	0.746	0.203
Agree	0.091	0.545	0.024	0.068	0.101	0.006
Neither ag~e	0.057	0.691	0.045	-0.171	0.218	0.022
Disagree	0.044	0.788	0.054	-0.373	0.674	0.080
Disagree s~y	0.015	0.852	0.071	-0.406	0.202	0.032
D						
Agree stro~y	0.017	0.782	0.039	0.333	0.285	0.025
Agree	0.067	0.126	0.012	-0.061	0.126	0.003
Neither ag~e	0.058	0.688	0.044	-0.106	0.087	0.009
Disagree	0.065	0.174	0.014	-0.061	0.103	0.003
Disagree s~y	0.043	0.869	0.034	0.196	0.288	0.022

	Dimension_2		
Categories	Coord	Sqcorr	Contrib
A			
Agree stro~y	0.176	0.103	0.018
Agree	−0.069	0.113	0.007
Neither ag~e	−0.289	0.786	0.084
Disagree	0.178	0.186	0.028
Disagree s~y	0.596	0.430	0.084
B			
Agree stro~y	0.331	0.131	0.038
Agree	−0.161	0.285	0.022
Neither ag~e	−0.233	0.690	0.055
Disagree	−0.068	0.065	0.006
Disagree s~y	0.509	0.526	0.179
C			
Agree stro~y	0.219	0.101	0.036
Agree	−0.143	0.444	0.032
Neither ag~e	−0.252	0.473	0.062
Disagree	0.153	0.114	0.018
Disagree s~y	0.728	0.650	0.136
D			
Agree stro~y	0.440	0.497	0.057
Agree	−0.002	0.000	0.000
Neither ag~e	−0.280	0.601	0.078
Disagree	−0.051	0.071	0.003
Disagree s~y	0.278	0.581	0.057

```
. mcaplot, overlay origin
```


MCA coordinate plot

Coordinates in principal normalization
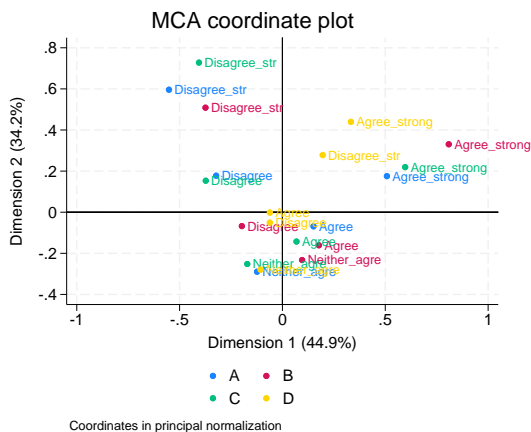
We displayed the origin axes in the plot with option `origin`. This plot makes it easier to see data associations. For more information on `mcaplot`, see [MV] **mca postestimation plots**.

A clear pattern is seen in the plot. Results from questions A, B, and C are clustered together, whereas question D forms a pattern of its own. Question D is formulated differently from A, B, and C, and the plot shows its incompatibility with the others.

Greenacre (2006, 70) produces this same plot. To obtain equivalent results, we reflect over the y axis with `xnegate` and adjust the scale of the graph with `xlabel()` and `ylabel()`.

```
. mcaplot, overlay xnegate origin ylabel(-1(.5)1.5) xlabel(-1.5(.5)1)
```
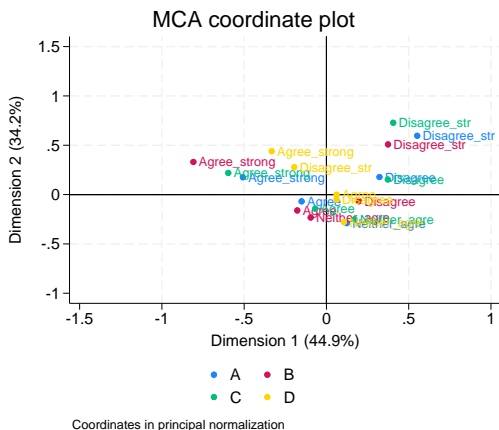

□ Technical note

The percentage of inertia in the top table of the `mca` output does not add up to 100%, although all singular values are displayed. Why? The percentages are lower-bound estimates in the Burt method with adjustments.

□

□ Technical note

Readers who are familiar with the discussion of the `normalize()` option in [MV] `ca` might be wondering why we are doing analysis of coordinates in the principal normalization. Principal normalization in `ca` does not allow good associations between the row and column categories, and so the symmetric normalization is the default used for `ca`. Principal normalization does allow studying the row categories or column categories separately from each other in `ca`. In `mca` there are only column categories. Consequently, the principal normalization is often preferred.

□

CA of the indicator matrix

▷ Example 3

We compare the previous result with that obtained using the `method(indicator)` option to perform an equivalent analysis to CA on the indicator matrix for these data. The first two dimensions explain only 22.2% of the total principal inertia.

```
. mca A-D, method(indicator)
```

Multiple/joint correspondence analysis		Number of obs	=	871
		Total inertia	=	4
Method: Indicator matrix		Number of axes	=	2

Dimension	Principal inertia	Percent	Cumul. percent
Dim 1	.4573792	11.43	11.43
Dim 2	.4309658	10.77	22.21
Dim 3	.3219257	8.05	30.26
Dim 4	.3064732	7.66	37.92
Dim 5	.2756747	6.89	44.81
Dim 6	.251928	6.30	51.11
Dim 7	.2425591	6.06	57.17
Dim 8	.2349506	5.87	63.05
Dim 9	.225468	5.64	68.68
Dim 10	.2206291	5.52	74.20
Dim 11	.2098376	5.25	79.44
Dim 12	.1971485	4.93	84.37
Dim 13	.1778833	4.45	88.82
Dim 14	.1691119	4.23	93.05
Dim 15	.1528191	3.82	96.87
Dim 16	.1252462	3.13	100.00
Total	4	100.00	

Statistics for column categories in standard normalization

		Overall			Dimension_1		
Categories	Mass	Quality	%inert		Coord	Sqcorr	Contrib
A							
Agree stro~y	0.034	0.280	0.054		1.837	0.244	0.078
Agree	0.092	0.100	0.039		0.546	0.080	0.019
Neither ag~e	0.059	0.218	0.048		-0.447	0.028	0.008
Disagree	0.051	0.220	0.050		-1.166	0.160	0.047
Disagree s~y	0.014	0.260	0.059		-1.995	0.106	0.037
B							
Agree stro~y	0.020	0.419	0.057		2.924	0.347	0.118
Agree	0.050	0.095	0.050		0.642	0.047	0.014
Neither ag~e	0.059	0.140	0.048		0.346	0.017	0.005
Disagree	0.081	0.127	0.042		-0.714	0.111	0.028
Disagree s~y	0.040	0.527	0.052		-1.354	0.161	0.050
C							
Agree stro~y	0.044	0.525	0.052		2.158	0.450	0.137
Agree	0.091	0.102	0.040		0.247	0.016	0.004
Neither ag~e	0.057	0.189	0.048		-0.619	0.051	0.015
Disagree	0.044	0.216	0.051		-1.349	0.179	0.054
Disagree s~y	0.015	0.312	0.059		-1.468	0.063	0.022
D							
Agree stro~y	0.017	0.155	0.058		1.204	0.049	0.017
Agree	0.067	0.008	0.046		-0.221	0.008	0.002
Neither ag~e	0.058	0.195	0.048		-0.385	0.020	0.006
Disagree	0.065	0.015	0.046		-0.222	0.008	0.002
Disagree s~y	0.043	0.168	0.052		0.708	0.048	0.015

	Dimension_2		
Categories	Coord	Sqcorr	Contrib
A			
Agree stro~y	0.727	0.036	0.012
Agree	−0.284	0.020	0.005
Neither ag~e	−1.199	0.190	0.055
Disagree	0.737	0.060	0.018
Disagree s~y	2.470	0.153	0.055
B			
Agree stro~y	1.370	0.072	0.025
Agree	−0.667	0.048	0.015
Neither ag~e	−0.964	0.123	0.036
Disagree	−0.280	0.016	0.004
Disagree s~y	2.108	0.367	0.117
C			
Agree stro~y	0.909	0.075	0.024
Agree	−0.592	0.086	0.021
Neither ag~e	−1.044	0.137	0.040
Disagree	0.635	0.037	0.012
Disagree s~y	3.017	0.249	0.089
D			
Agree stro~y	1.822	0.106	0.038
Agree	−0.007	0.000	0.000
Neither ag~e	−1.159	0.175	0.051
Disagree	−0.211	0.007	0.002
Disagree s~y	1.152	0.120	0.038

```
. mcaplot, overlay origin
```


MCA coordinate plot

Coordinates in standard normalization

Even though the first two dimensions explain only 22.2% of the total inertia, and difficulties exist in justifying the full χ^2 geometry for the CA of the indicator matrix, the same clear pattern we saw in the previous plot is seen here. Variables A, B, and C are related, but variable D does not belong with the rest.

◁

CA of the Burt matrix

▷ Example 4

Now we will look at results with method(burt) and the noadjust option. This performs a CA on the Burt matrix without adjusting the principal inertias. This model does slightly better than the indicator matrix approach, explaining 35.0% of the principal inertia in the first two dimensions. We display column points only for variable A to reduce the output.

```
. mca A-D, method(burt) noadjust points(A)
Multiple/joint correspondence analysis          Number of obs    =        871
                                                Total inertia    =  1.1276841
      Method: Burt/unadjusted inertias          Number of axes   =          2
```

Dimension	Principal inertia	Percent	Cumul. percent
Dim 1	.2091957	18.55	18.55
Dim 2	.1857315	16.47	35.02
Dim 3	.1036362	9.19	44.21
Dim 4	.0939258	8.33	52.54
Dim 5	.0759966	6.74	59.28
Dim 6	.0634677	5.63	64.91
Dim 7	.0588349	5.22	70.12
Dim 8	.0552018	4.90	75.02
Dim 9	.0508358	4.51	79.53
Dim 10	.0486772	4.32	83.84
Dim 11	.0440318	3.90	87.75
Dim 12	.0388675	3.45	91.20
Dim 13	.0316425	2.81	94.00
Dim 14	.0285988	2.54	96.54
Dim 15	.0233537	2.07	98.61
Dim 16	.0156866	1.39	100.00
Total	1.127684	100.00	

Statistics for column categories in standard normalization

		Overall			Dimension_1	
Categories	Mass	Quality	%inert	Coord	Sqcorr	Contrib
A						
Agree stro~y	0.034	0.445	0.055	1.837	0.391	0.115
Agree	0.092	0.169	0.038	0.546	0.136	0.028
Neither ag~e	0.059	0.344	0.047	−0.447	0.047	0.012
Disagree	0.051	0.350	0.050	−1.166	0.258	0.069
Disagree s~y	0.014	0.401	0.060	−1.995	0.170	0.055

	Dimension_2		
Categories	Coord	Sqcorr	Contrib
A			
Agree stro~y	0.727	0.054	0.018
Agree	−0.284	0.033	0.007
Neither ag~e	−1.199	0.298	0.084
Disagree	0.737	0.092	0.028
Disagree s~y	2.470	0.231	0.084

We do not provide a plot for this mca example; it would be the same as the previous one. MCA via the indicator matrix or the Burt matrix produces the same standard coordinates, although they produce different principal inertias. If normalize(principal) is used, different coordinates will be produced. Principal normalization relies on the principal inertias for normalization.

For more information on normalization, see either *Methods and formulas* in `mca` or the discussion of normalization in [MV] **ca**.

◁

Joint correspondence analysis

▷ Example 5

JCA attempts to remedy inflation of the total inertia by the block diagonal submatrices of the Burt matrix and is implemented as `method(joint)`. Results using JCA explain 90% of the principal inertia in the first two dimensions. With JCA, we must specify the dimensions we want in advance. The output contains principal inertias only for the first two dimensions (the default). For other methods, principal inertias for all dimensions are displayed. More dimensions can be requested with the `dim()` option. JCA is an iterative method, with repeated CAs and adjustments taking place until convergence is achieved.

We ask for coordinates in the principal normalization, `norm(principal)`, and ask for coordinates of only variable A to be displayed.

```
. mca A-D, method(joint) norm(principal) points(A)
```

Multiple/joint correspondence analysis   Number of obs   =   871
                                                Total inertia   = .18242477
    Method: Joint (JCA)   Number of axes =   2

Dimension	Principal inertia	Percent	Cumul. percent
Dim 1	.099091	54.32	54.32
Dim 2	.0650329	35.65	89.97
Total	.1824248	100.00	

Statistics for column categories in principal normalization

Categories	Mass	Overall Quality	%inert	Dimension_1 Coord	Sqcorr	Contrib
A						
Agree stro~y	0.034	0.964	0.052	-0.458	0.759	0.072
Agree	0.092	0.774	0.020	-0.169	0.733	0.027
Neither ag~e	0.059	0.884	0.030	0.048	0.025	0.001
Disagree	0.051	0.887	0.046	0.364	0.810	0.068
Disagree s~y	0.014	0.899	0.060	0.711	0.636	0.070

Categories	Dimension_2 Coord	Sqcorr	Contrib
A			
Agree stro~y	-0.238	0.205	0.030
Agree	0.040	0.042	0.002
Neither ag~e	0.281	0.859	0.071
Disagree	-0.112	0.077	0.010
Disagree s~y	-0.458	0.264	0.044

```
. mcaplot, overlay origin
```

MCA coordinate plot

Coordinates in principal normalization
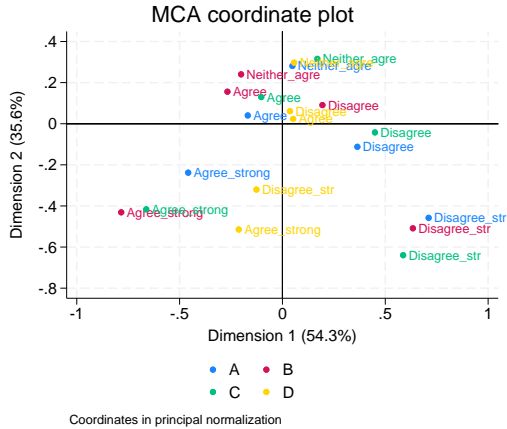
The plot shows the same relationships among the variables that we saw in the one after the indicator matrix approach to MCA. The main difference between the plots is the change in scale, partially because of the normalization and largely because of the change of method.

These same data are analyzed and plotted in Greenacre (2006, 66). To obtain an equivalent plot, we perform a reflection on the data with `ynegate`, add in the origin axes with `origin`, and use the same scale with options `xlabel()` and `ylabel()`,

```
. mcaplot, overlay ynegate origin ylabel(-1(.5)1.5) xlabel(-1.5(.5)1)
```

MCA coordinate plot

Coordinates in principal normalization
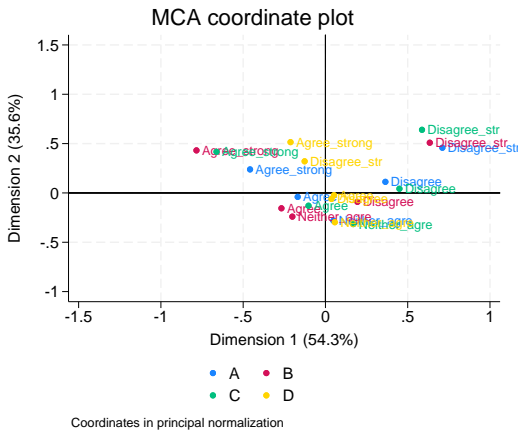
Note the similarities between this plot and the one obtained through the default approach to MCA via the Burt matrix and adjustments.

◁

Stored results

mca stores the following in e():

Scalars
  e(N)   number of observations
  e(f)   number of dimensions
  e(inertia)   total inertia
  e(ev_unique)   1 if all eigenvalues are distinct, 0 otherwise
  e(adjust)   1 if eigenvalues are adjusted, 0 otherwise (method(burt) only)
  e(converged)   1 if successful convergence, 0 otherwise (method(joint) only)
  e(iter)   number of iterations (method(joint) only)
  e(inertia_od)   proportion of off-diagonal inertia explained by extracted dimensions (method(joint) only)

Macros
  e(cmd)   mca
  e(cmdline)   command as typed
  e(names)   names of MCA variables (crossed or actual)
  e(supp)   names of supplementary variables
  e(defs)   per crossed variable: crossing variables separated by "\"
  e(missing)   missing, if specified
  e(crossed)   1 if there are crossed variables, 0 otherwise
  e(wtype)   weight type
  e(wexp)   weight expression
  e(title)   title in output
  e(method)   burt, indicator, or joint
  e(norm)   standard or principal
  e(properties)   nob noV eigen
  e(estat_cmd)   program used to implement estat
  e(predict)   program used to implement predict
  e(marginsnotok)   predictions disallowed by margins

Matrices
  e(Coding#)   row vector with coding of variable #
  e(A)   standard coordinates for column categories
  e(F)   principal coordinates for column categories
  e(cMass)   column mass
  e(cDist)   distance column to centroid
  e(cInertia)   column inertia
  e(cGS)   general statistics of column categories
                    [.,1] column mass
                    [.,2] overall quality
                    [.,3] inertia/sum(inertia)
                    [.,3*f+1] dim f: coordinate in e(norm) normalization
                    [.,3*f+2] dim f: contribution of the profiles to principal axes
                    [.,3*f+3] dim f: contribution of principal axes to profiles
                              (= squared correlation of profile and axes)
  e(rSCW)   weight matrix for row standard coordinates
  e(Ev)   principal inertias/eigenvalues
  e(inertia_e)   explained inertia (percent)
  e(Bmod)   modified Burt matrix of active variables (method(joint) only)
  e(inertia_sub)   variable-by-variable inertias (method(joint) only)

Functions
  e(sample)   marks estimation sample

Methods and formulas

Methods and formulas are presented under the following headings:

Notation

We use notation that is fairly standard in the literature on correspondence analysis (for example, Greenacre [2006]). Let $x_1, \ldots, x_q$ be categorical variables on N observations that are active in the analysis. To simplify notation, but without loss of generality, we assume that x_j is coded with consecutive integers $1, \ldots, n_j$. Let $\mathbf{Z}^{(j)}$ be the $N \times n_j$ binary indicator matrix associated with x_j, $\mathbf{Z}_{ih}^{(j)} = 1$ if and only if $x_{ij} = h$. Let

$$\mathbf{Z} = \left(\mathbf{Z}^{(1)}, \mathbf{Z}^{(2)}, \ldots, \mathbf{Z}^{(q)} \right)$$

be the $N \times J$ indicator matrix of the set of active x-variables, where $J = n_1 + \cdots + n_q$.

We will be consistent in letting i index observations $1, \ldots, N$, j index variables $1, \ldots, q$, and h index categories $1, \ldots, n_j$, or $1, \ldots, J$.

The $J \times J$ Burt matrix is defined as $\mathbf{B} = \mathbf{Z}'\mathbf{Z}$, or $\mathbf{B} = \mathbf{Z}'\mathbf{D}(w)\mathbf{Z}$, where w is the weight for the analysis and $\mathbf{D}(w)$ is a $J \times J$ square matrix with the weights on the diagonal and 0 off diagonal. The diagonal block of $\mathbf{B}$ associated with variable x_j is a diagonal matrix with the frequencies of x_j on the diagonal. The off-diagonal block of $\mathbf{B}$ associated with variables x_j and x_k is the two-way cross-tabulation of x_j and x_k.

In an analogous way, we define $\mathbf{B}^*$, the Burt matrix with more rows containing cross-tabulation from the supplementary variables. $\mathbf{B}^* = \mathbf{Z}^{*\prime}\mathbf{Z}$, where $\mathbf{Z}^*$ is the indicator matrix with more columns for the supplementary variables.

$\mathbf{D}(v)$, in general, represents a diagonal matrix with the elements of vector v on the diagonal and 0 off diagonal; $\mathbf{1}$ is a column vector of ones where length is defined by the context.

Using ca to compute MCA

The indicator approach to MCA involves an equivalent technique to performing a standard CA on the indicator matrix $\mathbf{Z}$; see Greenacre (2006). We refer to *Methods and formulas* in [MV] **ca** for details. The indicator approach could be handled by forming $\mathbf{Z}$ and invoking camat on $\mathbf{Z}$. If you had k categorical variables named v1, ..., vk, you could perform CA on the indicator matrix by taking the steps in the following code fragment:

```
tab v1, gen(V1_)
...
tab vk, gen(Vk_)
mkmat V1_* ... Vk_*, matrix(VALL)
camat VALL
```

CA of an indicator or Burt matrix

Z may be a rather large matrix: the number of rows is equal to the number of observations, and memory limitations would pose a limit for applications. Moreover, the interest in MCA is mostly on the column space, and so much computation would be wasted. Therefore, the implementation in mca exploits the close analogy between the indicator and Burt matrix approach for efficiency (see Greenacre [1984, chap. 5; 2006, app. A.3]). The consequence is that the CA results for the rows of **Z** are not computed. You may use the above scheme if you need row information.

Much literature (for example, Everitt and Dunn [2001]; Rencher and Christensen [2012]) identifies MCA with the application of CA on the Burt matrix $\mathbf{B} = \mathbf{Z}'\mathbf{Z}$.

This discussion is related to that of *Methods and formulas* in [MV] **ca**. We define

$$\mathbf{B}_{++} = \sum_{k=1}^{J} \sum_{h=1}^{J} \mathbf{B}_{kh}$$

$$\mathbf{P} = \mathbf{B}/\mathbf{B}_{++}$$

$$\mathbf{c} = \sum_{k=1}^{J} \mathbf{P}_{k*} = \mathbf{P}_{+*} = \mathbf{P}'\mathbf{1} \qquad \text{here } * \text{ represents all possible values}$$

$$\mathbf{S} = \mathbf{D}(\mathbf{c})^{-1/2}(\mathbf{P} - \mathbf{c}\mathbf{c}')\mathbf{D}(\mathbf{c})^{-1/2}$$

c is called the column mass. $\mathbf{D}(\mathbf{c})$ is the diagonal matrix with diagonal **c**; $\mathbf{D}(\mathbf{c})^{-1/2}$ is therefore the diagonal matrix with elements $1/\sqrt{c_t}$, where c_t is an element of **c**. **B** and **S** are symmetric. Thus the singular-value decomposition commonly associated with CA is equivalent to the spectral or eigen decomposition of **S**.

$$\mathbf{S} = \mathbf{V}\mathbf{\Phi}\mathbf{V}', \qquad \phi_1 \geq \phi_2 \geq \ldots$$

For identification purposes and to facilitate resampling methods, the eigenvectors of **V** are directed so that for h, $\mathbf{V}_{kh} > 0$ for the first k for which $\mathbf{V}_{kh} \neq 0$. The standard column coordinates **A** are the same for the indicator and the Burt approach and are defined as

$$\mathbf{A} = \mathbf{D}(\mathbf{c})^{-1}\mathbf{V}$$

$\mathbf{D}(\mathbf{c})^{-1}$ is the diagonal matrix with elements $1/c_t$, where c_t is an element of **c**.

In the indicator approach to MCA, the tth principal inertia is defined as $\lambda_t = \phi_t$, the total inertia as $\sum_t \lambda_t$. The inertia of column j or variable j, $\mathbf{In}^{(j)}$, is computed elementwise as

$$\mathrm{In}_h^{(j)} = \sum_{i=1}^{N} w_i \frac{(Z_{ih}^{(j)} - qc_h^{(j)})^2}{q^2 c_h^{(j)} w_+}$$

where $Z_{ih}^{(j)}$ is the i, hth element of the indicator matrix for variable j, w_i is the weight for observation i, q is the number of active variables, $c_h^{(j)}$ is the column mass of variable j for category h, and w_+ is the sum of the weights over the observations.

In the Burt approach to MCA, the unadjusted principal inertia is defined as $\lambda_t^{\text{unadj}} = \phi_t^2$, the total unadjusted inertia as $\sum_t \lambda_t^{\text{unadj}}$, and the unadjusted column inertias as $\mathbf{1}'\mathbf{S} \odot \mathbf{S}$, with $\odot$ the Hadamard or elementwise matrix product. The adjusted principal inertia, λ_t^{adj}, is defined as

$$\lambda_t^{\text{adj}} = \left(\frac{q}{q-1}\right)^2 \left(\phi_t^2 - \frac{1}{q}\right)^2 \quad \text{provided } q\phi_t > 1$$

The total inertia is defined as

$$\text{total inertia} = \left(\frac{q}{q-1}\right) \sum \phi_t^2 - \frac{(J-q)}{q^2}$$

The standard coordinates are independent of the principal inertia; with or without adjustment, these are defined as before

$$\mathbf{A} = \mathbf{D}(\mathbf{c})^{-1}\mathbf{V}$$

The principal coordinates $\mathbf{F}$ are defined as

$$\mathbf{F} = \mathbf{A}\mathbf{D}(\boldsymbol{\Lambda})^{1/2}$$

where $\boldsymbol{\Lambda}$ is a vector of adjusted or unadjusted principal inertias and $\mathbf{D}(\boldsymbol{\Lambda})^{1/2}$ is the diagonal matrix with elements $\lambda_t^{1/2}$ on the diagonals.

JCA

The implementation of JCA uses the alternating least-squares method proposed by Greenacre (1988, 2006). This algorithm forms a modification of the Burt matrix, changing the diagonal blocks associated with the variables, keeping the off-diagonal blocks unchanged. In the first iteration, $\mathbf{B}_0 = \mathbf{B}$. In iteration m, the blocks are replaced by the f-dimensional MCA approximation of the adjusted Burt matrix $\mathbf{B}_{m-1}$. Iteration continues until the change in the elements of $\mathbf{B}_m$ and $\mathbf{B}_{m-1}$ falls below a convergence tolerance. The JCA coordinates and inertias are computed from the converged solution, $\mathbf{B}_\infty$, analogous to the (adjusted) Burt method. The total inertia of the modified Burt matrix is defined as the sum of the inertias of the off-diagonal blocks.

To compute the f-dimensional MCA approximation of the adjusted Burt matrix $\mathbf{B}_{m-1}$, we perform MCA on $\mathbf{B}_{m-1}$ and then reconstruct the approximation of the data from the solution

$$\widehat{\mathbf{B}}_{hk} = \mathbf{B}_{++}c_h c_k \left(1 + \sum_{t=1}^{f} \phi_t^2 A_{ht} A_{kt}\right)$$

where A_{ht} is an element of the standard coordinate matrix $\mathbf{A}$, c_h and c_k are the column masses, and ϕ_t are the eigenvalues as in the computation of the CA of the Burt matrix. We then update the main diagonal submatrices of $\mathbf{B}_{m-1}$ with the corresponding entries of $\widehat{\mathbf{B}}$ to obtain $\mathbf{B}_m$.

Supplementary variables

The coordinates of supplementary variables are computed as weighted averages of the column coordinates by using the so-called CA transition formulas. As outlined by Greenacre (2006), standard coordinates may be used for averaging, with the profiles of the indicator representation of supplementary columns as weights. Supplementary principal column coordinates are computed as weighted averages of the standard active column coordinates, and then supplementary standard coordinates are computed by division by the principal inertias.

To compute, we add the supplementary variables to the Burt matrix as more rows; if $\mathbf{B}$ is the Burt matrix of the active variables then $\mathbf{B}^*$ is the Burt matrix with the additional cross-tabulation from the supplementary variables. Define $\mathbf{P}$ as above and $\mathbf{P}^*$ analogously with the supplementary variables added. MCA is performed on $\mathbf{B}$ as before, and information from this solution is then applied to $\mathbf{B}^*$. Let p^* represent the elements of $\mathbf{P}^*$. Let k index categories of supplementary variables, h index categories of active variables, and t index dimensions of the solution. Let $\mathbf{A}$ be the standard coordinates of the active variables, as computed previously. Then the principal coordinate for category k and dimension s is computed as:

$$g_{kt} = \sum_{h=1}^{J} \frac{p_{hk}^*}{p_{+k}^*} A_{ht}$$

Coordinates in standard coordinates are obtained by division by the square root of the corresponding principal inertia as described above.

predict

predict after mca produces variables with the MCA coordinates as displayed by mca for both active and supplementary variables. Formulas are shown above. predict can also compute row coordinates also known as row scores. Row coordinates computed are always based on the indicator method. The standard row coordinate for the tth dimension for the ith observation with indicator matrix elements Z_{ih} is computed as

$$R_{it} = \sum_{h=1}^{J} \frac{Z_{ih} A_{ht}}{q \sqrt{\phi_t}}$$

where $\mathbf{A}$ is the matrix of standard coordinates, q is the number of active variables in the analysis, and ϕ_t is an eigenvalue of the CA on the Burt matrix. To get the row coordinate in principal normalization, one multiplies by the square root of the corresponding principal inertia.

References

Everitt, B. S., and G. Dunn. 2001. *Applied Multivariate Data Analysis*. 2nd ed. London: Arnold.

Gower, J. C. 1990. Fisher's optimal scores and multiple correspondence analysis. *Biometrics* 46: 947–961. https://doi.org/10.2307/2532440.

———. 2006. Divided by a common language—analyzing and visualizing two-way arrays. In *Multiple Correspondence Analysis and Related Methods*, ed. M. J. Greenacre and J. Blasius. Boca Raton, FL: Chapman and Hall/CRC.

Gower, J. C., and D. J. Hand. 1996. *Biplots*. London: Chapman and Hall.

Greenacre, M. J. 1984. *Theory and Applications of Correspondence Analysis*. London: Academic Press.

———. 1988. Correspondence analysis of multivariate categorical data by weighted least-squares. *Biometrika* 75: 457–467. https://doi.org/10.2307/2336595.

———. 2006. From simple to multiple correspondence analysis. In *Multiple Correspondence Analysis and Related Methods*, ed. M. J. Greenacre and J. Blasius. Boca Raton, FL: Chapman and Hall.

Greenacre, M. J., and J. Blasius, ed. 1994. *Correspondence Analysis in the Social Sciences*. London: Academic Press.

———. 2006. *Multiple Correspondence Analysis and Related Methods*. Boca Raton, FL: Chapman and Hall.

ISSP. 1993. International Social Survey Programme: Environment. http://www.issp.org.

Milan, L., and J. C. Whittaker. 1995. Application of the parametric bootstrap to models that incorporate a singular value decomposition. *Applied Statistics* 44: 31–49. https://doi.org/10.2307/2986193.

Rencher, A. C., and W. F. Christensen. 2012. *Methods of Multivariate Analysis*. 3rd ed. Hoboken, NJ: Wiley.

Also see